

A Note on the Gauge Group of the Electroweak Interactions

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We propose a three-fold covering of the group $U(2)$ as a gauge group for the electroweak interactions for the purpose of describing fields with integer and fractional electric charges with respect to the residual electromagnetic gauge group after a spontaneous breaking of the gauge symmetry. In a more general scheme we construct a three-fold covering of $U(n)$ and consider for the case $n = 2$ several representations which are used in the construction of a model of the electroweak interactions in a subsequent paper.

PACS number(s): 02.20.-a; 11.15.-q; 12.15.-y

I. INTRODUCTION

In the Weinberg-Salam (WS) model [1,2] the transformation laws under the group $U(1)$ of the weak hypercharge Y are different for the quark and lepton fields. Among the irreducible representations of $U(1)$, namely $e^{i\alpha} \rightarrow e^{in\alpha}$, $0 \leq \alpha \leq 2\pi$ and $n \in \mathbb{Z}$ any integer, for the left and right lepton fields one takes the transformations (for each generation)

$$\psi_L \rightarrow e^{-i\alpha} \psi_L, \quad \psi_R \rightarrow e^{-2i\alpha} \psi_R \quad (1)$$

(or $Y = -1$ for ψ_L , and $Y = -2$ for ψ_R), in order to obtain the correct electric charges of the leptons (in units of the elementary charge e). With the same purpose one sets for the quark fields

$$u_L \rightarrow e^{\frac{i\alpha}{3}} u_L, \quad d_L \rightarrow e^{\frac{i\alpha}{3}} d_L, \quad u_R \rightarrow e^{\frac{4i\alpha}{3}} u_R, \quad d_R \rightarrow e^{\frac{-2i\alpha}{3}} d_R, \quad etc \dots, \quad (2)$$

i.e. $Y = 1/3$ for u_L , d_L and $Y = 4/3$ for u_R , $Y = -2/3$ for d_R . Obviously, these formulae do not fit with the irreducible representations of the group $U(1)$ defined as

$$U(1) = \{e^{i\alpha} \mid 0 \leq \alpha \leq 2\pi\}. \quad (3)$$

In this paper we propose to use as a gauge group for the WS model a three-fold covering of $U(2) = (U(1) \times SU(2))/\mathbb{Z}_2$ in order to deal with descent representations on the fields. We apply the term metaunitary group for this three-fold covering and denote it by $MU(2)$. In a recent publication Roepstorff and Vehns [3] propose a subgroup G of $SU(5)$ as a gauge group for the *standard model* such that G appears also as a covering of $U(2)$. Our construction for the group $MU(2)$ is motivated by an argument of Guillemin and Sternberg [4] for the definition of a two-fold covering L of the general linear group, aimed to define a representation of the type $g \rightarrow \det^{1/2} g$ of L . Noting that the Lie algebras of the groups $U(1)$ and \mathbb{R} coincide, we may expect that for the description of fields with electric charge proportional to $e/3$, a suitable group may be a factor group of $\mathbb{R} \times SU(2)$. In order to give a more general framework, we present in Section II the construction of a three-fold covering $MU(n)$ of the group $U(n)$. In Section III we specialize to the case $n = 2$ and consider several representations of $MU(2)$ and its Lie algebra which will turn useful for the description of leptons and quarks in a subsequent paper.

II. THE METAUNITARY GROUP $\text{MU}(N)$

As mentioned above, we are looking for a gauge group of the electroweak interactions as a suitable factor group of $\mathbb{R} \times \text{SU}(2)$. Following an argument from [4] and in order to provide a more general framework, we begin with $\mathbb{R} \times \text{SU}(n)$, $n \geq 2$, where the group composition law reads

$$(u, A) \cdot (v, B) = (u + v, AB), \quad u, v \in \mathbb{R}, \quad A, B \in \text{SU}(n) \quad (4)$$

and consider the subgroup of $\mathbb{R} \times \text{SU}(n)$ with elements

$$\left\{ \left(k \frac{2\pi}{n}, e^{-k \frac{2\pi i}{n}} I \right) \mid k \in \mathbb{Z} \right\}, \quad (5)$$

which is isomorphic to \mathbb{Z} . This subgroup is a normal one and through the map $T : \mathbb{R} \times \text{SU}(n) \rightarrow \text{U}(n)$, defined as

$$T(u, A) = e^{iu} A, \quad (6)$$

we obtain an isomorphism $(\mathbb{R} \times \text{SU}(n))/\mathbb{Z} = \text{U}(n)$. Indeed

$$T\left(u + k \frac{2\pi}{n}, e^{-k \frac{2\pi i}{n}} A\right) = T(u, A). \quad (7)$$

The factor group $\text{MU}(n) = (\mathbb{R} \times \text{SU}(n))/3\mathbb{Z}$, consisting of the equivalence classes

$$[u, A] = \left\{ \left(u + 3k \frac{2\pi}{n}, e^{-3k \frac{2\pi i}{n}} A \right) \mid k \in \mathbb{Z} \right\}, \quad (8)$$

we call *metaunitary group*. Clearly the map $T : \text{MU}(n) \rightarrow \text{U}(n)$ defines a homomorphism onto $\text{U}(n)$. Moreover, T defines a three-fold covering of $\text{U}(n)$. Indeed, the kernel of T as a map acting on $\text{MU}(n)$ consists of the elements

$$[0, I], \quad \left[\frac{2\pi}{n}, e^{-\frac{2\pi i}{n}} I \right] \quad \text{and} \quad \left[\frac{4\pi}{n}, e^{-\frac{4\pi i}{n}} I \right]. \quad (9)$$

Certainly, the group $\text{MU}(n)$ is locally isomorphic to $\text{U}(n)$ and $\text{SU}(n) \times \text{U}(1)$. The same technique is applicable for constructing arbitrary l -fold covering of $\text{U}(n)$.

III. PARTICULAR REPRESENTATIONS OF $\text{MU}(2)$ AND ITS LIE ALGEBRA

A. Representations of $\text{MU}(2)$

We here specialize to the case $n = 2$. Consider the map $\text{Det}^{\frac{1}{3}} : \mathbb{R} \times \text{SU}(2) \rightarrow \text{U}(1)$ defined by

$$\text{Det}^{\frac{1}{3}}(u, A) = e^{\frac{2iu}{3}}, \quad (u, A) \in \mathbb{R} \times \text{SU}(2). \quad (10)$$

Due to the property

$$\text{Det}^{\frac{1}{3}}(u + 3k\pi, e^{-3k\pi i} A) = \text{Det}^{\frac{1}{3}}(u, A) \quad (11)$$

the map $\text{Det}^{\frac{1}{3}}$ is well defined on $\text{MU}(2)$ and gives a homomorphism of $\text{MU}(2)$ onto $\text{U}(1)$. For every integer k the mapping $\text{Det}^{\frac{k}{3}} : \text{MU}(2) \rightarrow \text{U}(1)$ given by

$$\text{Det}^{\frac{k}{3}}[u, A] = \left(\text{Det}^{\frac{1}{3}}[u, A] \right)^k, \quad (12)$$

is also a homomorphism of $\text{MU}(2)$ on $\text{U}(1)$. For $k = 3$

$$\text{Det}[u, A] = \det \circ T[u, A] \quad (13)$$

where “det” stands for the usual determinant. Using the maps T and $\text{Det}^{\frac{k}{3}}$ we define the homomorphisms $T^k : \text{MU}(2) \rightarrow \text{U}(2)$ by

$$T^k [u, A] = \text{Det}^{\frac{k}{3}} [u, A] T [u, A] = e^{iu(1+\frac{2k}{3})} A . \quad (14)$$

Some particular cases are

$$T^0 [u, a] = T [u, A] = e^{iu} A , \quad T^{-2} [u, A] = e^{-\frac{iu}{3}} A , \quad (15)$$

$$\text{Det}^{\frac{1}{3}} [u, A] = e^{\frac{2iu}{3}} , \quad \text{Det}^{-\frac{2}{3}} [u, A] = e^{-\frac{4iu}{3}} , \quad \text{Det} [u, A] = e^{2iu} . \quad (16)$$

The one-parameter subgroup of $\text{MU}(2)$,

$$\text{MU}_{\text{em}}(1) = \left\{ \left[-\frac{\alpha}{2}, \begin{pmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{\frac{i\alpha}{2}} \end{pmatrix} \right] \in \text{MU}(2) \mid \alpha \in \mathbb{R} \right\} , \quad (17)$$

has the meaning of the group generated by the electric charge generator

$$Q = \frac{1}{2} Y + I_3 = \frac{1}{2} I + \frac{1}{2} \sigma_3 , \quad (18)$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The image of Q in each representation of $\text{MU}_{\text{em}}(1)$ in a space V is identified with the electric charge in this representation, the eigenvalues q_i of Q being identified with the charges of the corresponding eigenvectors from V .

Let e_1, e_2 be a basis in \mathbb{C}^2 . The second exterior degree $\Lambda^2 \mathbb{C}^2$ is a one-dimensional complex space generated by $e_1 \wedge e_2$ which carries the representations $\text{Det}^{\frac{k}{3}}$ for different k . Let $a = u e_1 + v e_2 \in \mathbb{C}^2$ and $w e_1 \wedge e_2 \in \Lambda^2 \mathbb{C}^2$. Using the notation $\mathcal{M}(\alpha)$, $0 \leq \alpha \leq 2\pi$, for $\text{MU}_{\text{em}}(1)$, one finds in the representations (15,16)

$$1) \quad T [\mathcal{M}(\alpha)] = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} , \quad Q_T = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} , \quad q_u = 0 , \quad q_v = -1 . \quad (19)$$

$$2) \quad T^{-2} [\mathcal{M}(\alpha)] = \begin{pmatrix} e^{\frac{2i\alpha}{3}} & 0 \\ 0 & e^{-\frac{i\alpha}{3}} \end{pmatrix} , \quad Q_{T^{-2}} = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} , \quad q_u = \frac{2}{3} , \quad q_v = -\frac{1}{3} . \quad (20)$$

$$3) \quad \text{Det}^{\frac{1}{3}} [\mathcal{M}(\alpha)] = e^{-\frac{i\alpha}{3}} , \quad Q_{\text{Det}^{\frac{1}{3}}} = -\frac{1}{3} , \quad q_w = -\frac{1}{3} . \quad (21)$$

$$4) \quad \text{Det}^{-\frac{2}{3}} [\mathcal{M}(\alpha)] = e^{\frac{2i\alpha}{3}} , \quad Q_{\text{Det}^{-\frac{2}{3}}} = \frac{2}{3} , \quad q_w = \frac{2}{3} . \quad (22)$$

$$5) \quad \text{Det} [\mathcal{M}(\alpha)] = e^{-i\alpha} , \quad Q_{\text{Det}} = -1 , \quad q_w = -1 . \quad (23)$$

The choice of these representations is justified by the reduction of $\text{MU}(2)$ to $\text{MU}_{\text{em}}(1)$ in analogy with the reduction of $\text{SU}(2) \times \text{U}(1)$ to $\text{U}_{\text{em}}(1)$ in the WS model. Then a direct sum of one-dimensional representations of $\text{MU}_{\text{em}}(1)$ appears, each of them with a fixed electric charge.

B. Representations of the Lie Algebra of $\text{MU}(2)$

The groups $\text{MU}(2)$ and $\mathbb{R} \times \text{SU}(2)$ are locally isomorphic and one has for their Lie algebras

$$\text{Lie MU}(2) = \mathbb{R} \oplus \text{Lie SU}(2) . \quad (24)$$

Accordingly, a set of four generators for $\text{MU}(2)$ is given by

$$X^a = \left(0, \frac{\sigma^a}{2}\right), \quad a = 1, 2, 3, \text{ and } \quad X = \left(-\frac{1}{2}, 0\right), \quad (25)$$

where σ^a are the Pauli matrices and

$$[X^a, X^b] = i\varepsilon^{abc}X^c, \quad [X^a, X] = 0.$$

The subgroups of MU(2), generated by X^a and X , are

$$G_{X^a}(t) = \left\{ \left[0, e^{\frac{i\sigma^a}{2}t}\right] \mid t \in \mathbb{R} \right\}, \quad a = 1, 2, 3, \quad (26)$$

$$G_X(t) = \left\{ \left[-\frac{1}{2}t, I\right] \mid t \in \mathbb{R} \right\}. \quad (27)$$

Each representation T of the group MU(2) generates a representation T_* of its Lie algebra. For the particular representations (19)-(23) defined in the previous subsection one finds for the generators X^a and X

$$1) \quad T \left[0, e^{\frac{i\sigma^a}{2}t}\right] = e^{\frac{i\sigma^a}{2}t}, \quad T \left[-\frac{1}{2}t, I\right] = e^{-\frac{i}{2}t}I. \quad (28)$$

$$T_*(X^a) = -i \frac{d}{dt} e^{\frac{i\sigma^a}{2}t} \Big|_{t=0} = \frac{\sigma^a}{2}. \quad (29)$$

$$T_*(X) = -i \frac{d}{dt} e^{-\frac{i}{2}t}I \Big|_{t=0} = -\frac{I}{2}. \quad (30)$$

$$2) \quad T^{-2} \left[0, e^{\frac{i\sigma^a}{2}t}\right] = e^{\frac{i\sigma^a}{2}t}, \quad T^{-2} \left[-\frac{1}{2}t, I\right] = e^{\frac{i}{6}t}I. \quad (31)$$

$$T_*^{-2}(X^a) = -i \frac{d}{dt} e^{\frac{i\sigma^a}{2}t} \Big|_{t=0} = \frac{\sigma^a}{2}. \quad (32)$$

$$T_*^{-2}(X) = -i \frac{d}{dt} e^{-\frac{i}{2}t}I \Big|_{t=0} = \frac{I}{6}. \quad (33)$$

$$3) \quad \text{Det}^{\frac{1}{3}} \left[0, e^{\frac{i\sigma^a}{2}t}\right] = 1, \quad \text{Det}^{\frac{1}{3}} \left[-\frac{1}{2}t, I\right] = e^{-\frac{i}{3}t}. \quad (34)$$

$$\text{Det}_*^{\frac{1}{3}}(X^a) = -i \frac{d}{dt} 1 \Big|_{t=0} = 0. \quad (35)$$

$$\text{Det}_*^{\frac{1}{3}}(X) = -i \frac{d}{dt} e^{-\frac{i}{3}t} \Big|_{t=0} = -\frac{1}{3}. \quad (36)$$

$$4) \quad \text{Det}^{-\frac{2}{3}} \left[0, e^{\frac{i\sigma^a}{2}t}\right] = 1, \quad \text{Det}^{-\frac{2}{3}} \left[-\frac{1}{2}t, I\right] = e^{\frac{2i}{3}t}. \quad (37)$$

$$\text{Det}_*^{-\frac{2}{3}}(X^a) = -i \frac{d}{dt} 1 \Big|_{t=0} = 0. \quad (38)$$

$$\text{Det}_*^{-\frac{2}{3}}(X) = -i \frac{d}{dt} e^{\frac{2i}{3}t} \Big|_{t=0} = \frac{2}{3}. \quad (39)$$

$$5) \quad \text{Det} \left[0, e^{\frac{i\sigma^a}{2}t}\right] = 1, \quad \text{Det} \left[-\frac{1}{2}t, I\right] = e^{-it}. \quad (40)$$

$$\text{Det}_*(X^a) = -i \frac{d}{dt} 1 \Big|_{t=0} = 0. \quad (41)$$

$$\text{Det}_*(X) = -i \frac{d}{dt} e^{-it} \Big|_{t=0} = -1. \quad (42)$$

These representations of *Lie* MU(2) will be used in a subsequent paper for the explicit form of the covariant derivatives of the fields in a model based on the gauge group MU(2).

ACKNOWLEDGMENTS

One of us (V.R.) is grateful to Prof. R. Kerner and the Laboratoire de Physique des Particules, Université Pierre et Marie Curie, Paris, for their hospitality during a visit when this work was started. This work was partly supported (I.V.) by the DOE Research Grant under Contract No. De-FG-02-93ER-40764.

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